LOOPS AND SEMIDIRECT PRODUCTS

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1. Introduction

A *left loop* $(B, \cdot)$ is a set B together with a binary operation $\cdot$ such that (i) for each $a \in B$, the left translation mapping $L_a : B \to B$ defined by $L_a(x) = a \cdot x$ is a bijection, and (ii) there exists a two-sided identity $1 \in B$ satisfying $1 \cdot x = x \cdot 1 = x$ for every $x \in B$. A right loop is similarly defined, and a *loop* is both a right loop and a left loop [4] [5].

In this paper we study semidirect products of loops with groups. This is a generalization of the familiar semidirect product of groups. Recall that if G is a group with subgroups B and H where B is normal, $G = BH$, and $B \cap H = \{e\}$, then G is said to be an *internal* semidirect product of B with H. On the other hand, if B and H are groups and $\sigma : H \to \mathrm{Aut}(B) : h \mapsto \sigma_h$ is a homomorphism, then the *external* semidirect product of B with H given by σ, denoted $B \rtimes_\sigma H$, is the set $B \times H$ with the multiplication

$$(a, h)(b, k) = (a \cdot \sigma_h(b), hk). \tag{1.1}$$

A special case of this is the *standard* semidirect product where H is a subgroup of the automorphism group of B, and σ is the inclusion mapping. The relationship between internal, external and standard semidirect products is well known.

These considerations can be generalized to loops. We now describe the contents of the sequel.

In §2, we consider the natural embedding of a left loop B into its permutation group $\mathrm{Sym}(B)$. This leads to a factorization of $\mathrm{Sym}(B)$ into a sub*set* $\hat{B}$ consisting of the left translations of B and a sub*group* $\mathrm{Rot}(B)$ consisting of permutations fixing the identity element. We then discuss left inner mappings and deviations [23], and show how these characterize those permutations which are pseudo-automorphisms and automorphisms. After introducing classes of loops which will be discussed throughout the paper, we then consider how the aforementioned factorization of $\mathrm{Sym}(B)$ is related to the group multiplication. Here the left inner mappings and deviations play a role in decomposing the product of permutations. This leads us to describing those subgroups of $\mathrm{Rot}(B)$ which respect the given factorization of $\mathrm{Sym}(B)$. Finally, we give Sabinin's definition of the *standard* semidirect product of a left loop B with (certain types of) subgroups of $\mathrm{Rot}(B)$ [23]. This semidirect product has occasionally been rediscovered for various classes of loops. We conclude the section with an example.

In §3, we consider *internal* semidirect products of left loops and groups: given a group G, a subgroup $H < G$, and a transversal $B \subseteq G$ of H which contains the

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identity, B naturally has the structure of a left loop. This is equivalent to putting a loop structure on the set G/H of cosets, but it is closer to the examples to work with subsets of G. We consider the relationship between the loop structure of B and the multiplication in G, paralleling the discussion in §2. This first part of the discussion follows Sabinin [23], but with considerably more detail. We then consider conditions under which subgroups and factor groups inherit the semidirect product structure. A particular case of the latter is obtained by modding out the kernel of the natural action of H on B (an action which generalizes conjugation in the group case). We then introduce a collection of properties of the decomposition $G = BH$ which imply (and under certain conditions are equivalent to) the loop identities discussed in §2. This part of our study is related to work of Ungar [25] and Kreuzer and Wefelscheid [21], but we do not assume as much structure at the outset. We then give examples of internal semidirect products, illustrating some of the results of the section.

In §4, we generalize the standard semidirect product of a left loop B with a particular subgroup of $\mathrm{Rot}(B)$ to an *external* semidirect product of a left loop B with a group H. As for the usual semidirect product of groups, the main interest here is in the case where the defining homomorphism from H to $\mathrm{Rot}(B)$ is not injective. Our construction seems to be new, and we give examples. We also discuss how the three semidirect products are related, generalizing the relationship between the usual standard, internal, and external semidirect products of groups.

There exist notions of semidirect products of loops which are different from that which we consider here. One definition is as follows: a loop R is an internal semidirect product of the normal subloop P by the subloop Q if $R = PQ$ and $P \cap Q = \{1\}$. This definition was given by Birkenmeier *et al* [2] and Birkenmeier and Xiao [3], who studied nonassociative loops which are internal semidirect products of groups. Goodaire and Robinson [10] defined an internal semidirect product similarly with additional conditions given in terms of associators. In contrast, our internal semidirect product follows Sabinin [23]: $G = BH$ is a factorization of a group G into a subgroup H and a transversal B. Even if B with its induced operation turns out to be a group, it is not necessarily a subgroup of G. Also, H does not necessarily stabilize B by conjugation. Thus the two notions of semidirect product are quite distinct.

In group theory, the question of which groups have a semidirect product structure is answered by cohomology theory. Cohomology has been generalized to loops in at least two distinct ways; see Eilenberg and MacLane [6], and Johnson and Leedham-Green [14]. At present, we do not know if the semidirect product of the present paper has a suitable cohomological interpretation.

2. Standard Semidirect Products

Let $\mathrm{Sym}(B)$ denote the group of permutations of the left loop B. A permutation $\phi \in \mathrm{Sym}(B)$ is *rotary* [13] if $\phi(1) = 1$. Let $\mathrm{Rot}(B)$ denote the subgroup of all rotary permutations. Let $\hat{B} = \{L_a : a \in B\}$ be the set of left translations, and let $\mathcal{LM}(B) = \langle \hat{B} \rangle$ be the *left multiplication group*, which is the subgroup of $\mathrm{Sym}(B)$ generated by $\hat{B}$. Let $\mathcal{LM}_1(B) = \mathcal{LM}(B) \cap \mathrm{Rot}(B)$.

The mapping $B \to \hat{B} : a \longmapsto L_a$ is injective; indeed, $L_a = L_b$ implies $a = L_a(1) = L_b(1) = b$. Note that $\hat{B}$ itself can be given the structure of a left loop isomorphic to B with the obvious definition:

$$L_a \cdot L_b = L_{a\cdot b} \tag{2.1}$$

for $a, b \in B$.

Let G be any group G satisfying $\mathcal{LM}(B) \leq G \leq \mathrm{Sym}(B)$, and let $H = G \cap \mathrm{Rot}(B)$. For any $\phi \in G$, we have $\phi = L_a \circ \psi$ where $a = \phi(1)$ and $\psi = L_a^{-1} \circ \phi$. Clearly ψ is rotary, and thus $\psi \in H$ since G contains $\mathcal{LM}(B)$. The factorization of ϕ into a left translation L_a in $\hat{B}$ and a rotary permutation ψ in H is unique. Indeed, if $L_a \circ \psi = L_b \circ \varphi$ for $a, b \in B$, $\psi, \varphi \in H$, then applying both sides to 1 gives $a = b$, and thus $L_a \circ \psi = L_a \circ \varphi$; cancelling L_a gives $\psi = \varphi$.

Summarizing, for any group G satisfying $\mathcal{LM}(B) \leq G \leq \mathrm{Sym}(B)$, we have the following decomposition:

$$G = \hat{B}H \tag{2.2}$$

where $H = G \cap \mathrm{Rot}(B)$. The factorization of elements is unique, and we also have

$$\hat{B} \cap H = \{\iota\} \tag{2.3}$$

where ι denotes the identity mapping on B.

For $a, b \in B$, the permutation $\lambda_{a,b} \in \mathrm{Sym}(B)$ defined by

$$\lambda_{a,b} = L_{ab}^{-1} \circ L_a \circ L_b \tag{2.4}$$

is called a *left inner mapping*. (Our notation is a slight variant of the usual one [4] [5].) Let $\mathcal{LI}(B)$ denote the *left inner mapping group*, which is the subgroup of $\mathcal{LM}(B)$ generated by the left inner mappings. (This is also known as the "left associant" of B [23].) The following proposition collects some properties of left inner mappings, and also shows how the factorization (2.2) can be used to characterize properties of B.

Proposition 2.1. *Let B be a left loop.*

1. *For $a, b \in B$, $\lambda_{a,b}$ is rotary, and hence*

$$\mathcal{LI}(B) \leq \mathcal{LM}_1(B). \tag{2.5}$$

2. *For all $a \in B$,*

$$\lambda_{1,a} = \lambda_{a,1} = \iota. \tag{2.6}$$

3. *The following are equivalent.*
   (a) *B is a group, i.e., $L_a \circ L_b = L_{a\cdot b}$ for all $a, b \in B$.*
   (b) *For all $a, b \in B$, $L_a \circ L_b \in \hat{B}$.*
   (c) *For all $a, b \in B$, $\lambda_{a,b} = \iota$.*

Proof. Parts (1) and (2) are obvious consequences of the definition. For (3), that (a) implies (b) is clear. For all $a, b \in B$, we have $L_a \circ L_b = L_{a\cdot b} \circ \lambda_{a,b}$ which shows the equivalence of (a) and (c). If (b) holds, then by uniqueness of the factorization, we have $L_a \circ L_b = L_{a\cdot b}$ and thus (a) holds. ∎

A permutation $\phi \in \mathrm{Sym}(B)$ is called a *pseudo-automorphism* with *companion* $c \in B$ if $c \cdot \phi(xy) = (c \cdot \phi(x))\phi(y)$ for all $x, y \in B$. A pseudo-automorphism is rotary (take $y = 1$ and cancel c). The set $\mathrm{psAut}(B)$ of pseudo-automorphisms of B is a group under composition of mappings, and is thus a subgroup of $\mathrm{Rot}(B)$. A pseudo-automorphism with companion 1 is an *automorphism* of B. Let $\mathrm{Aut}(B)$ denote the group of automorphisms of B.

For each $a \in B$ and $\phi \in \mathrm{Sym}(B)$, the permutation $\mu_a(\phi) \in \mathrm{Sym}(B)$ defined by

$$\mu_a(\phi) = L_{\phi(a)}^{-1} \circ \phi \circ L_a \circ \phi^{-1} \tag{2.7}$$

is called the *deviation* of ϕ at a [23]. As the next result shows, deviations measure how much arbitrary permutations "deviate" from being (pseudo-)automorphisms.

Proposition 2.2. *Let B be a left loop.*

1. *$\mu_a(\phi)$ is rotary for all $a \in B$ if and only if ϕ is rotary.*
2. *For all $a \in B$, $\phi \in \mathrm{Rot}(B)$,*

$$\mu_1(\phi) = \iota \tag{2.8}$$
$$\mu_a(\iota) = \iota. \tag{2.9}$$

3. *Let $\phi \in \mathrm{Sym}(B)$ be given. The following are equivalent.*
   (a) *ϕ is a pseudo-automorphism with companion $c \in B$.*
   (b) *For all $x \in B$, $L_c \circ \phi \circ L_x \circ \phi^{-1}$ is a left translation.*
   (c) *For all $x \in B$, $\mu_x(\phi) = \lambda_{c,\phi(x)}^{-1}$.*
4. *Let $\phi \in \mathrm{Sym}(B)$ be given. The following are equivalent.*
   (a) *ϕ is an automorphism.*
   (b) *For all $x \in B$, $\phi \circ L_x \circ \phi^{-1}$ is a left translation.*
   (c) *For all $a \in B$, $\mu_a(\phi) = \iota$.*

Proof. Parts (1) and (2) are obvious consequences of the definition.
3. That (a) implies (b) is trivial. For $x \in B$, we have

$$L_c \circ \phi \circ L_x \circ \phi^{-1} = L_c \circ L_{\phi(x)} \circ \mu_x(\phi) = L_{c\cdot\phi(x)} \circ \lambda_{c,\phi(x)} \circ \mu_x(\phi).$$

This shows the equivalence of (a) and (c). If (b) holds, then the uniqueness of the factorization implies $L_c \circ \phi \circ L_x \circ \phi^{-1} = L_{c\cdot\phi(x)}$ and thus (a) holds.
4. This follows immediately from (3). ∎

Next we review the definitions of the classes of loops we will consider in more detail in §3, and also in our discussion of rotary-closed groups below.

For $a \in B$, define $a' \in B$ by $a \cdot a' = 1$. A left loop B is said to satisfy the *left inverse property* (LIP) if either of the following equivalent identities hold: for all $a \in B$,

$$L_a^{-1} = L_{a'} \tag{2.10}$$
$$\lambda_{a,a'} = \iota. \tag{2.11}$$

for all $a \in B$. This implies a' is a necessarily a (unique) two-sided inverse of $a \in B$.

A left loop B is said to satisfy the *left alternative property* (LAP) if either of the following equivalent identities hold: for all $a \in B$,

$$L_a^2 = L_{a\cdot a} \tag{2.12}$$
$$\lambda_{a,a} = \iota. \tag{2.13}$$

for all $a \in B$.

A left loop B is said to be a (left) *Bol loop* if it satisfies either of the following equivalent identities: for $a, b \in B$,

$$L_a \circ L_b \circ L_a = L_{a\cdot(b\cdot a)} \tag{2.14}$$
$$\lambda_{a,b\cdot a} \circ \lambda_{b,a} = \iota. \tag{2.15}$$

A Bol loop satisfies LIP (take $a = b'$ in (2.14)), LAP (take $b = 1$ in (2.14)), and is also a right loop (the unique solution of $x \cdot a = b$ is $x = a'(ab \cdot a')$).

A left loop B is said to be an *LC* left loop if it satisfies either of the following equivalent identities: for $a, b \in B$,

$$L_a \circ L_a \circ L_b = L_{a\cdot(a\cdot b)} \tag{2.16}$$

$$\lambda_{a,a\cdot b} \circ \lambda_{a,b} = \iota. \tag{2.17}$$

An LC left loop satisfies LIP (take $b = a'$ in (2.16)) and LAP (take $b = 1$ in (2.16)).

A left loop B is said to be a *pseudo-A_l left loop* if every left inner mapping is a pseudo-automorphism, or equivalently, if $\mathcal{LI}(B) \leq \mathrm{psAut}(B)$. A pseudo-$A_l$ left loop with LIP is said to be *pseudo-homogeneous*. For example, if B is a Bol loop, then for $a, b \in B$, $\lambda_{a,b}$ is a pseudo-automorphism with companion $(ab)(a'b')$ [10]. Thus every Bol loop is a pseudo-homogeneous loop.

A left loop B is said to be an *A_l left loop* if every left inner mapping is an automorphism, or equivalently, if $\mathcal{LI}(B) \leq \mathrm{Aut}(B)$. An A_l left loop with LIP is said to be *homogeneous* [17]. A homogeneous Bol loop is sometimes called a "gyrogroup", as defined by Ungar in [28]. The equivalence of gyrogroups and homogeneous Bol loops was noted by Ròzga [22].

A left loop B is said to satisfy the *automorphic inverse property* (AIP) if $(x\cdot y)' = x' \cdot y'$ for all $x, y \in B$. If B has two-sided inverses, then the inversion mapping $J : B \to B : x \mapsto x'$ is a rotary permutation of B, and is also an involution, i.e., $J^2 = \iota$. In this case, AIP is equivalent to the identity

$$J \circ L_a \circ J = L_{a'} \tag{2.18}$$

for all $a \in B$. In a homogeneous left loop or a Bol loop, AIP is equivalent to the *Bruck identity*

$$a \cdot (b \cdot (b \cdot a)) = (a \cdot b) \cdot (a \cdot b), \tag{2.19}$$

$a, b \in B$. Homogeneous left loops satisfying AIP are called *symmetric* [17]. A Bol loop satisfying AIP is called a *Bruck loop*. Bruck loops are also known as "K-loops" [15] [16] [20] [25], "gyrogroups" [26], and later, as "gyrocommutative gyrogroups" [28]. The equivalence between Bruck loops and K-loops was shown by Kreuzer [20]. The equivalence between Bruck loops and gyrocommutative gyrogroups was noted by Ròzga [22]. A Bruck loop is automatically A_l, and is thus a homogeneous Bol loop [10]. A Bruck loop B is called a *B-loop* if the mapping $a \mapsto a \cdot a$ is a permutation of B.

Next we give a simple example of how the unique factorization of permutations can be used to characterize a loop identity. In §3, we give similar results in the general setting of group factorizations. The example we present here does not have an generalization appropriate for that section. Recall that for a left loop B with two-sided inverses, we denote the inversion mapping by $Jx = x'$, $x \in B$.

Proposition 2.3. *If B has two-sided inverses, then B satisfies AIP if and only if, for all $a \in B$, $J \circ L_a \circ J \in \hat{B}$.*

Proof. We have $J \circ L_a \circ J = L_{a'} \circ \mu_a(J)$ since $J^2 = \iota$. If the left side is a translation, then by uniqueness of the factorization, it must be equal to $L_{a'}$. ∎

It is natural to ask how the factorization (2.2) of a group into a sub*set* with a loop structure and a sub*group* interacts with the group multiplication. We first examine this question for $G = \mathrm{Sym}(B)$ and $H = \mathrm{Rot}(B)$. For permutations $L_a \circ \phi$

and $L_b \circ \psi$ with $a, b \in B$, $\phi, \psi \in \mathrm{Rot}(B)$, we have $(L_a \circ \phi \circ L_b \circ \psi)(1) = a \cdot \phi(b)$. Also, $L_{a\cdot\phi(b)}^{-1} \circ L_a \circ \phi \circ L_b = \lambda_{a,\phi(b)} \circ \mu_b(\phi) \circ \phi$. Put together, these observations give the following factorization of a product in $\mathrm{Sym}(B) = \hat{B}\ \mathrm{Rot}(B)$.

Proposition 2.4. *Let B be a left loop. For all $a, b \in B$, $\phi, \psi \in \mathrm{Rot}(B)$,*

$$(L_a \circ \phi) \circ (L_b \circ \psi) = L_{a\cdot\phi(b)} \circ (\lambda_{a,\phi(b)} \circ \mu_b(\phi) \circ \phi \circ \psi). \tag{2.20}$$

Using (2.1), (2.20) can be rewritten as

$$(L_a \circ \phi) \circ (L_b \circ \psi) = (L_a \cdot L_{\phi(b)}) \circ (\lambda_{a,\phi(b)} \circ \mu_b(\phi) \circ \phi \circ \psi).$$

Thus we see that the operation (2.1) is simply the projection of the composition $L_a \circ L_b$ onto $\hat{B}$.

For the factorization (2.20) to hold in a subgroup G of $\mathrm{Sym}(B)$, it is clear that it is necessary that the rotary part of (2.20) be in the subgroup $H = G \cap \mathrm{Rot}(B)$. A subgroup $H \le \mathrm{Rot}(B)$ is said to be *rotary-closed* if $\lambda_{a,b} \in H$ for every $a, b \in B$ and if $\mu_a(\phi) \in H$ for every $a \in B$, $\phi \in H$. (This term is adapted from [13]. Sabinin [23] called a rotary-closed group a "transassociant" of B.)

Let $\mathcal{N}_0 = \mathcal{LI}(B)$, and for $j \ge 0$, let $\mathcal{N}_{j+1} = \langle \mathcal{N}_j, \{\mu_a(\phi) : a \in B, \phi \in \mathcal{N}_j\}\rangle$ be the group generated by $\mathcal{N}_j$ and by the set of all deviations of elements of $\mathcal{N}_j$. For $j \ge 0$, we have $\mathcal{N}_j \le \mathcal{N}_{j+1}$, and clearly each element of $\mathcal{N}_j$ is rotary. Therefore $\mathcal{N} := \bigcup_{j\ge 0} \mathcal{N}_j$ is a subgroup of $\mathcal{LM}_1(B)$.

Lemma 2.5. *If B is a left loop with LIP, then $\mathcal{LI}(B) = \mathcal{N} = \mathcal{LM}_1(B)$.*

Proof. Indeed, assume $\phi \in \mathcal{LM}_1(B)$. Using (2.10), we have $\phi = L_{a_1} \circ L_{a_2} \circ \cdots \circ L_{a_n}$ for some $a_1, \ldots, a_n \in B$. By an easy induction,

$$\phi = L_{a_1(a_2(\cdots a_n)\cdots)} \circ \lambda_{a_1,a_2(\cdots a_n)} \circ \cdots \circ \lambda_{a_{n-1},a_n}.$$

Since ϕ is rotary, the uniqueness of the factorization implies that $L_{a_1(a_2(\cdots a_n)\cdots)}$ is rotary, and hence $L_{a_1(a_2(\cdots a_n)\cdots)} = \iota$. Thus $\phi \in \mathcal{LI}(B)$. Together with (2.5), this establishes the result. ∎

Proposition 2.6. *Let B be a left loop.*

1. *$\mathcal{N}$ is a rotary-closed group.*
2. *If H is a rotary-closed group, then $\mathcal{N} \le H$.*
3. *If H is a group satisfying $\mathcal{LM}_1(B) \le H \le \mathrm{Rot}(B)$, then H is rotary-closed.*
4. *If B is a left loop with LIP and H is a rotary-closed group, then $\mathcal{LM}_1(B) \le H \le \mathrm{Rot}(B)$.*
5. *If B is a pseudo-A_l left loop, and if H is a group satisfying $\mathcal{LI}(B) \le H \le \mathrm{psAut}(B)$, then H is rotary-closed.*
6. *If B is an A_l left loop, and if H is a group satisfying $\mathcal{LI}(B) \le H \le \mathrm{Aut}(B)$, then H is rotary-closed.*

Proof. (1) and (2) are consequences of the definitions.

3. This follows from (2.5), (2.7) and Proposition 2.2(1).
4. This follows from (2) and Lemma 2.5.
5. This follows from Proposition 2.2(3).
6. This follows from Proposition 2.2(4). ∎

Proposition 2.6(3) describes the class of rotary-closed groups with which we motivated the factorization (2.2). In arbitrary left loops, there might be rotary-closed groups H satisfying $\mathcal{N} \leq H \leq \mathrm{Rot}(B)$, but such that H does not contain $\mathcal{LM}_1(B)$.

Consider again the case where $G = \mathrm{Sym}(B) = \hat{B}\ \mathrm{Rot}(B)$. The factorization (2.2), along with (2.3), gives a one-to-one correspondence between $B \times \mathrm{Rot}(B)$ and $\mathrm{Sym}(B)$ given by $(a, \phi) \longmapsto L_a \circ \phi$. Thus we may use Proposition 2.4 to define a binary operation on $B \times \mathrm{Rot}(B)$:

$$(a, \phi) \cdot (b, \psi) = (a \cdot \phi(b), \lambda_{a,\phi(b)} \circ \mu_b(\phi) \circ \phi \circ \psi) \tag{2.21}$$

for $a, b \in B$, $\phi, \psi \in \mathrm{Rot}(B)$. By construction, $(B \times \mathrm{Rot}(B), \cdot)$ is a group isomorphic to $\mathrm{Sym}(B)$. We now present Sabinin's definition [23] of the semidirect product of a left loop with one of its rotary-closed groups.

Definition 2.7. Let B be a left loop and let $H \leq \mathrm{Rot}(B)$ be a rotary-closed group. Define a binary operation $\cdot$ on the set $B \times H$ as follows:

$$(a, \phi) \cdot (b, \psi) = (a \cdot \phi(b), \lambda_{a,\phi(b)} \circ \mu_b(\phi) \circ \phi \circ \psi)$$

for all $a, b \in B$, $\phi, \psi \in H$. Then $(B \times H, \cdot)$ is called the *standard semidirect product* of B with H, and is denoted $B \rtimes H$.

It is immediate from this definition that $B \rtimes H$ is a subgroup of the group $B \rtimes \mathrm{Rot}(B) \cong \mathrm{Sym}(B)$. We have proven the following result.

Proposition 2.8. *(*[23]*, Thm. 2) Let B be a left loop and let $H \leq \mathrm{Rot}(B)$ be a rotary-closed subgroup. Then $B \rtimes H$ is a group.*

Incidentally, as Sabinin [23] has noted, in order for $B \rtimes H$ to be a group, it is only necessary for B to have a *right* identity element 1. In this case, (2.3) becomes $\hat{B} \cap H = \{L_1\}$ because $L_1 \neq \iota$. The identity permutation ι factors as $\iota = L_1 \circ L_1^{-1}$ with $L_1^{-1} \in H$.

We now consider some special cases.

Remark 2.9.

1. If B is a group, then the product in $B \rtimes H$ is given by

$$(a, \phi) \cdot (b, \psi) = (a \cdot \phi(b), \mu_b(\phi) \circ \phi \circ \psi).$$

   This generalized semidirect product of groups was rediscovered by Jajcay [13], who dubbed it the "rotary" product of groups. The semidirect product $\mathrm{Sym}(B) \cong B \rtimes \mathrm{Rot}(B)$ can be seen as a detailed description of the algebraic structure of the regular representation of B.
2. Assume B is a pseudo-A_l left loop and that $\mathcal{LI}(B) \leq H \leq \mathrm{psAut}(B)$. By Proposition 2.6(5), H is rotary-closed. In this case, the product in $B \rtimes H$ is given by

$$(a, \phi) \cdot (b, \psi) = (a \cdot \phi(b), \lambda_{a,\phi(b)} \circ \lambda^{-1}_{c,\phi(b)} \circ \phi \circ \psi),$$

   $a, b \in B$, $\phi, \psi \in H$, where c is a companion of ϕ, using Proposition 2.2(3). The semidirect product group $\mathrm{psAff}(B) \cong B \rtimes \mathrm{psAut}(B)$ is called the *pseudo-affine group* of B.
3. Assume B is an A_l left loop and that $\mathcal{LI}(B) \leq H \leq \mathrm{Aut}(B)$. By Proposition 2.6(6), H is rotary-closed. In this case, the product in $B \rtimes H$ is given by

$$(a, \phi) \cdot (b, \psi) = (a \cdot \phi(b), \lambda_{a,\phi(b)} \circ \phi \circ \psi),$$

$a, b \in B$, $\phi, \psi \in H$, using Proposition 2.2(4). The semidirect product group $\mathrm{Aff}(B) \cong B \rtimes \mathrm{Aut}(B)$ is called the *affine group* of B. For homogeneous left loops, this semidirect product was rediscovered by Kikkawa [17] and later, using different terminology, by Ungar [25].

4. If B is a group and H is a subgroup of $\mathrm{Aut}(B)$, then $B \rtimes H$ is the usual standard semidirect product of groups.

We now give an explicit example of a standard semidirect product.

Example 2.10. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the complex unit disk. Note that the circle group $S^1 = \{a \in \mathbb{C} : |a| = 1\}$ acts on $\mathbb{D}$ by multiplication of complex numbers. For $x, y \in \mathbb{D}$, define

$$x \oplus y = \frac{x + y}{1 + \bar{x}y}. \tag{2.22}$$

Then $(\mathbb{D}, \oplus)$ is a B-loop [18] [27]. The left inner mappings are given by unimodular complex numbers

$$\lambda_{x,y}(z) = \frac{1 + x\bar{y}}{1 + \bar{x}y} z \tag{2.23}$$

for $x, y, z \in \mathbb{D}$. If we identify S^1 with its natural image in $\mathrm{Aut}(\mathbb{D})$, then $\mathrm{Aut}(\mathbb{D})$ is generated by S^1 and the complex conjugation mapping $x \longmapsto \bar{x}$. The semidirect product $\mathbb{D} \rtimes S^1$ turns out to be isomorphic to the orientation-preserving Möbius group of the disk $\mathbb{D}$. The semidirect product $\mathbb{D} \rtimes \mathrm{Aut}(\mathbb{D})$ is isomorphic to the full Möbius group of $\mathbb{D}$.

3. Internal Semidirect Products

Let G be a group with identity element e, let H be a subgroup of G, and let B be a *transversal* of H in G, i.e., for every $g \in G$, there exists a unique $a \in B$ and a unique $h \in H$ such that $g = ah$. (Equivalently, each element of B is a representative of a unique left coset of H in G.) We will call the factorization $G = BH$ a *transversal decomposition*. (Sabinin [23] calls B a "quasi-reductant". Here we adapt more standard group-theoretic terminology.) Given $g \in G$, we will denote the unique factors (projections) of g in B and H by $[g]_B$ and $[h]_H$, respectively; thus $g = [g]_B[g]_H$. Note the relations

$$[gh]_B = [g]_B \tag{3.1}$$

$$[gh]_H = [g]_H h \tag{3.2}$$

for all $g \in G$, $h \in H$. (We will use (3.1)-(3.2) frequently without directly referring to them.)

Let $1 = [e]_B$. Then we have

$$B \cap H = \{1\}. \tag{3.3}$$

Indeed, $[1^2]_B = [1e]_B = [1]_B = 1$. This implies $1^2 = 1[1^2]_H$, and hence $1 = [1^2]_H$. Thus $1 \in B \cap H$. On the other hand, if $g \in B \cap H$, then $g = [g]_B = [e]_B = 1$.

Now let $h_e = [e]_H$ and let $\tilde{B} = Bh_e = \{ah_e : a \in B\}$. Then $\tilde{B}$ is a transversal of H in G. Indeed, for $g \in G$, we have $g = ([g]_B h_e)(h_e^{-1}[g]_H)$. To show uniqueness, suppose $g = ak$ for $a \in \tilde{B}$, $k \in H$. Then $[g]_B h_e = [a]_B h_e = ah_e^{-1}h_e = a$ and thus $h_e^{-1}[g]_H = k$. In addition, we clearly have $\tilde{B} \cap H = \{e\}$.

This discussion shows that there is no loss in assuming that $e = 1$, i.e.,

$$B \cap H = \{e\}. \tag{3.4}$$

In this case, B is called a *unital* transversal and $G = BH$ is said called a *unital* transversal decomposition. (B is sometimes called a "uniform quasi-reductant" [23], and $G = BH$ is called an "exact" decomposition [19] [21].) We will assume throughout that our transversal decompositions are unital without specifically mentioning it.

Let $G = BH$ be a transversal decomposition. Define a binary operation on B as follows: for $a, b \in B$, let

$$a \cdot b = [ab]_B. \tag{3.5}$$

Also define

$$l(a, b) = [ab]_H. \tag{3.6}$$

We call $l : B \times B \to H$ the *transversal mapping.* Note that

$$1 \cdot a = a \cdot 1 = a \tag{3.7}$$

and

$$l(1, a) = l(a, 1) = e \tag{3.8}$$

for all $a \in B$.

The following result is well-known ([23], Thm. 7; [21], Thm. 3.2).

Proposition 3.1. *$(B, \cdot)$ is a left loop.*

Proof. From (3.7), we have that 1 is a two-sided identity. For $a, b \in B$, we compute $a \cdot [a^{-1}b]_B = [a[a^{-1}b]_B]_B = [aa^{-1}b]_B = b$. Thus each left translation $L_a : B \to B : x \longmapsto a \cdot x$ has an inverse given by $L_a^{-1}(x) = [a^{-1}x]_B$. ∎

Obviously $(B, \cdot)$ is a subgroup of G if and only if the transversal mapping $l : B \times B \to H$ is trivial, i.e., $l(a, b) = e$ for all $a, b \in B$.

For $a \in B$, recall that a' is the unique right inverse of a, i.e., $a \cdot a' = 1$. We compute

$$1e = aa^{-1} = a[a^{-1}]_B[a^{-1}]_H = (a \cdot [a^{-1}]_B)l(a, [a^{-1}]_B)[a^{-1}]_H.$$

Matching B- and H-components of both sides, we conclude

$$a' = [a^{-1}]_B \tag{3.9}$$

$$l(a, a')^{-1} = [a^{-1}]_H. \tag{3.10}$$

Having considered the components of a product of elements of B, we now do likewise for arbitrary $ah, bk \in G$ ($a, b \in B$, $h, k \in H$). We have

$$ahbk = a[hbh^{-1}]_B[hbh^{-1}]_H hk.$$

This leads us to the following definitions. For $a \in B$, $h \in H$, define

$$\sigma_h(a) = [hah^{-1}]_B = [ha]_B \tag{3.11}$$

and

$$m(a, h) = [hah^{-1}]_H = [ha]_H h^{-1}. \tag{3.12}$$

The following is an immediate consequence of these definitions.

Proposition 3.2. *For all $a \in B$, $h \in H$,*

$$\sigma_h(1) = 1 \tag{3.13}$$

$$m(a,e) = e \tag{3.14}$$

$$m(1,h) = e. \tag{3.15}$$

Theorem 3.3. 1. *The mapping $\sigma : H \to \mathrm{Rot}(B)$ defined by $h \longmapsto \sigma_h$ is a homomorphism.*

2. *For all $a, b \in B$, $h \in H$,*

$$\lambda_{a,b} = \sigma_{l(a,b)} \tag{3.16}$$

$$\mu_a(\sigma_h) = \sigma_{m(a,h)}. \tag{3.17}$$

Proof. 1. Note that each σ_h is invertible with $\sigma_h^{-1} = \sigma_{h^{-1}}$. Thus each σ_h is a permutation and (3.13) shows σ_h is rotary. For $h, k \in H$, $a \in B$, we compute

$$\begin{aligned} \sigma_h \circ \sigma_k(a) &= [h[ka]_B]_B \\ &= [hka]_B \\ &= \sigma_{hk}(a). \end{aligned}$$

This establishes the result.

2. For $a, b, c \in B$, we have

$$\begin{aligned} \lambda_{a,b}(c) &= (L_{a\cdot b}^{-1} \circ L_a \circ L_b)(c)a \\ &= [[a \cdot b]_B^{-1}[a[bc]_B]_B]_B \\ &= [[ab]_B^{-1}abc]_B \\ &= [[ab]_H c]_B \\ &= \sigma_{l(a,b)}(c). \end{aligned}$$

This establishes (3.16). Next, for $a, b \in B$, $h \in H$, we compute

$$\begin{aligned} \sigma_{m(a,h)}(b) &= [m(a,h)b]_B \\ &= [[ha]_H h^{-1} bh]_B \\ &= [[ha]_B^{-1} ha\sigma_h^{-1}(b)]_B \\ &= [\sigma_h(a)^{-1}\sigma_h(a\sigma_h^{-1}(b))]_B \\ &= \left(L_{\sigma_h(a)}^{-1} \circ \sigma_h \circ L_a \circ \sigma_h^{-1}\right)(b) \\ &= \mu_a(\sigma_h)(b). \end{aligned}$$

This establishes (3.17). ∎

Corollary 3.4. *The group $\sigma(H) \leq \mathrm{Rot}(B)$ is rotary-closed.*

We thus conclude the discussion which motivated (3.11) and (3.12).

Proposition 3.5. *For all $a, b \in B$, $h, k \in K$,*

$$ahbk = (a \cdot \sigma_h(b))l(a, \sigma_h(b))m(b,h)hk \tag{3.18}$$

Comparison of Proposition 3.5 with Proposition 2.4 suggests the following.

Definition 3.6. Let $(B, \cdot)$ be the left loop induced by a transversal decomposition $G = BH$. Then we say that G is an *internal semidirect product* of $(B, \cdot)$ with H.

Remark 3.7.
1. If $l : B \times B \to H$ is trivial, but $m : B \times H \to H$ is nontrivial, then $(B, \cdot)$ is a subgroup of G, and $G = BH$ is the internal version of Jajcay's "rotary product" [13] of subgroups.
2. If $m : B \times H \to H$ is trivial, but $l : B \times B \to H$ is nontrivial, then the product of $ah, bk \in G$ simplifies to
$$(ah)(bk) = (a \cdot \sigma_h(b))l(a, b)hk.$$
As will be shown below, in this case $(B, \cdot)$ is an A_l left loop, and $G = BH$ is the internal version of the semidirect product rediscovered (for homogeneous left loops) by Kikkawa [17] and Ungar [25].
3. Both $l : B \times B \to H$ and $m : B \times H \to H$ are trivial if and only if B is a normal subgroup of G. In this case, $G = BH$ is the usual internal semidirect product of subgroups.
4. Another case where $(B, \cdot)$ is a group is if $\sigma : H \to \mathrm{Rot}(B)$ is trivial, i.e., $\sigma(H) = \{\iota\}$; this follows from (3.16) and Proposition 2.1(3). However, if the transversal mapping $l : B \times B \to H$ is nontrivial, then B is not a subgroup of G, and if $m : B \times H \to H$ is nontrivial, then H does not normalize B.

We now consider the inheritance of internal semidirect product structure by subgroups. Let $G = BH$ be a transversal decomposition giving an internal semidirect product of $(B, \cdot)$ by H. Assume that G_1 is a subgroup of G and let $B_1 = B \cap G_1$ and $H_1 = H \cap G_1$. For $g \in G_1$, we clearly have $[g]_B \in B_1$ if and only if $[g]_H \in H_1$. When either of these conditions hold, we say that G_1 *respects* the transversal decomposition of G, or equivalently, that G_1 respects the internal semidirect product structure of G. If G_1 respects $G = BH$, then $G_1 = B_1H_1$ is itself a transversal decomposition, which means that G_1 is an internal semidirect product of B_1 with H_1. In particular, the operation $\cdot$ on B restricts to B_1, which shows that the left loop $(B_1, \cdot)$ is a subloop of $(B, \cdot)$. Finally, if G_1 and G_2 are both subgroups respecting $G = BH$, then clearly the intersection $G_1 \cap G_2$ satisfies this property as well.

Next we consider the inheritance of internal semidirect product structure by factor groups. Let $G = BH$ be a transversal decomposition and let $K \lhd H$ be a normal subgroup of G. An arbitrary element gK of G/K factors as $gK = (ah)K = (aK)(hK)$ where $aK \in BK = \{aK : a \in B\}$ and $hK \in H/K$. This factorization is clearly unique. Also, $BK \cap H/K = \{K\}$. Thus
$$G/K = BK \cdot H/K$$
is a transversal decomposition of the factor group G/K. Denote the induced binary operation (3.5) by $\cdot_K : BK \times BK \to BK$ and the induced transversal mapping by $l_K : BK \times BK \to H/K$.

Since $B \cap K = \{\iota\}$, the set BK can be identified with B itself. Thus we compare two factorizations of products. For $a, b \in B$, we have
$$(aK)\,(bK) = (aK \cdot_K bK)\, l_K(aK, bK),$$
and also
$$\begin{aligned} (aK)\,(bK) &= (ab)K = (a \cdot b)\, l(a, b)K \\ &= ((a \cdot b)\, K)\, (l(a, b)K)\, . \end{aligned}$$
By uniqueness, we have
$$\begin{aligned} aK \cdot_K bK &= (a \cdot b)\, K \\ l_K(aK, bK) &= l(a, b)K \end{aligned}$$

for all $a, b \in B$. It follows that under the mapping $a \mapsto aK$, the left loop $(B, \cdot)$ induced by the transversal decomposition $G = BH$ is isomorphic to the left loop $(BK, \cdot_K)$ induced by the transversal decomposition $G/K = BK \cdot H/K$. Making this identification, we may think of

$$G/K = B \cdot H/K \tag{3.19}$$

as being a transversal decomposition of G/K.

We now consider a specific case of factor group inheritance of internal semidirect product structure. Let $G = BH$ be an internal semidirect product of the left loop $(B, \cdot)$ with the group H. By Corollary 3.4, we may also form the standard semidirect product $B \rtimes \sigma(H)$. The homomorphism $\sigma : H \to \sigma(H)$ naturally extends to a mapping $\hat{\sigma} : G \to B \rtimes \sigma(H)$ given by $\hat{\sigma}(ah) = (a, \sigma_h)$. This mapping is trivially surjective, and it is also a homomorphism. Indeed, for $a, b \in B$, $h, k \in H$, we compute

$$\begin{aligned}
\hat{\sigma}(ah)\hat{\sigma}(bk) &= (a, \sigma_h)(b, \sigma_k) \\
&= (a \cdot \sigma_h(b), \lambda_{a,\sigma_h(b)} \circ \mu_a(\sigma_h) \circ \sigma_h \circ \sigma_k) \\
&= (a \cdot \sigma_h(b), \sigma_{l(a,\sigma_h(b))m(a,h)hk}) \\
&= \hat{\sigma}((a \cdot \sigma_h(b))l(a, \sigma_h(b))m(a, h)hk) \\
&= \hat{\sigma}(ahbk).
\end{aligned}$$

The kernel of $\hat{\sigma}$ is the subgroup $\ker(\hat{\sigma}) = \ker(\sigma)$. It follows that the exact sequence of groups

$$e \to \ker(\sigma) \to H \to \sigma(H) \to e \tag{3.20}$$

induces an exact sequence of internal semidirect product groups

$$e \to \ker(\hat{\sigma}) \to G \to B \rtimes \sigma(H) \to e. \tag{3.21}$$

Let $G = BH$ be a transversal decomposition. The preceding discussion shows that $\ker(\sigma)$, which is a normal subgroup of H, is also a normal subgroup of G. Of course, the exactness of (3.20) and (3.21) imply the isomorphisms $H/\ker(\sigma) \cong \sigma(H)$ and $G/\ker(\sigma) \cong B \rtimes \sigma(H)$. On the other hand, our earlier discussion of normal subgroups leading up to (3.19) gives us

$$G/\ker(\sigma) = B \cdot H/\ker(\sigma) \tag{3.22}$$

as a transversal decomposition of $G/\ker(\sigma)$. Note also the obvious isomorphism of groups $B \cdot H/\ker(\sigma) \to B \rtimes \sigma(H)$ given by $a(h \ker(\sigma)) \mapsto (a, \sigma_h)$.

Remark 3.8. Consider again the special case of Remark 3.7(4) where $\sigma(H) = \{\iota\}$. Then making the usual identifications, (3.21) simplifies to

$$e \to H \to G \to B \to e. \tag{3.23}$$

As noted, if l is nontrivial, then B is a group, but not a subgroup. Instead, we see from (3.23) that B is an isomorphic copy of the factor group G/H.

A transversal decomposition $G = BH$ is said to be *reduced* if the homomorphism $\sigma : H \to \mathrm{Rot}(B)$ defined by (3.11) is injective. (This term is adapted from [7].) If $G = BH$ is a reduced transversal decomposition, then obviously the elements of the image $l(B, B)$ of the transversal map are in a one-to-one correspondence with the left inner mappings (by (3.16)). For a transversal decomposition $G = BH$, let $H_0 = \langle l(B, B)\rangle$ denote the subgroup of H generated by the image of the transversal mapping. If $G = BH$ is reduced, then the restriction of σ to H_0 is an isomorphism onto the left inner mapping group $\mathcal{LI}(B)$.

Let $G = BH$ be a transversal decomposition. For $B \subseteq G$, let $B^2 = \{a^2 : a \in B\}$ and let $B^{-1} = \{a^{-1} : a \in B\}$. The following list of properties will turn out to imply, and in the reduced case be equivalent to, certain loop identities.

(G-LIP)	$B^{-1} \subseteq B$.
(G-LAP)	$B^2 \subseteq B$.
(G-Bol)	For all $a \in B$, $aBa \subseteq B$.
(G-LC)	For all $a \in B$, $a^2 B \subseteq B$.
(G-psA$_l$)	For each $h \in H$, there exists $c \in B$ such that $chBh^{-1} \subseteq B$.
(G-psA$_l$)$_0$	For each $h \in H_0$, there exists $c \in B$ such that $chBh^{-1} \subseteq B$.
(G-A$_l$)	For all $h \in H$, $hBh^{-1} \subseteq B$.
(G-A$_l$)$_0$	For all $h \in H_0$, $hBh^{-1} \subseteq B$.
(G-Br)	For all $a, b \in B$, $ab^2a = (a \cdot b)^2$.

Lemma 3.9. *Let $G = BH$ be a transversal decomposition, and let $h \in H$ be given.*

1. *The following are equivalent.*
   (a) *There exists $c \in B$ such that $m(a,h) = l(c, \sigma_h(a))^{-1}$ for all $a \in B$.*
   (b) *There exists $c \in B$ such that $chBh^{-1} \subseteq B$.*
2. *The following are equivalent.*
   (a) *$m : B \times B \to H$ is trivial, i.e., $m(a,h) = e$ for all $a \in B$, $h \in H$.*
   (b) *$hBh^{-1} \subseteq B$.*

Proof. 1. For $a \in B$, we have

$$chah^{-1} = c\sigma_h(a)m(a,h) = (c \cdot \sigma_h(a))l(c, \sigma_h(a))m(a,h).$$

From this, the equivalence is clear.

2. This is obvious from the definition of $m : B \times H \to H$. ∎

Theorem 3.10. *Let $G = BH$ be a transversal decomposition.*

1. *If (G-LIP) holds, then $(B, \cdot)$ satisfies LIP. If $G = BH$ is reduced and $(B, \cdot)$ satisfies LIP, then (G-LIP) holds.*
2. *If (G-LAP) holds, then $(B, \cdot)$ satisfies LAP. If $G = BH$ is reduced and $(B, \cdot)$ satisfies LAP, then (G-LAP) holds.*
3. *If (G-Bol) holds, then $(B, \cdot)$ is a Bol loop. If $G = BH$ is reduced and $(B, \cdot)$ is a Bol loop, then (G-Bol) holds.*
4. *If (G-LC) holds, then $(B, \cdot)$ is an LC left loop. If $G = BH$ is reduced and $(B, \cdot)$ is an LC left loop, then (G-LC) holds.*
5. *If (G-psA$_l$) holds, then $(B, \cdot)$ is a pseudo-A_l left loop. If $G = BH$ is reduced and $(B, \cdot)$ is a pseudo-A_l left loop, then (G-psA$_l$)$_0$ holds.*
6. *If (G-A$_l$) holds, then $(B, \cdot)$ is an A_l left loop. If $G = BH$ is reduced and $(B, \cdot)$ is an A_l left loop, then (G-A$_l$)$_0$ holds.*

Proof. 1. Assume (G-LIP) holds. For $a \in B$, we have $1e = aa^{-1} = (a \cdot a^{-1})l(a, a^{-1})$. Matching components, we have $a' = a^{-1}$ and $l(a, a^{-1}) = e$. By (3.16), $\lambda_{a^{-1},a} = \iota$, which is (2.11). Conversely, if $G = BH$ is reduced and (2.11) holds, then $\sigma_{l(a,a')} = \sigma_e$ implies $aa' = l(a, a') = e$. Thus $a^{-1} = a' \in B$.

2. Assume (G-LAP) holds. For $a \in B$, we have $a^2 = (a \cdot a)l(a, a) \in B$, and thus $l(a, a) = e$. By (3.16), $\lambda_{a,a} = \iota$, which is (2.13). Conversely, if $G = BH$ is reduced and (2.13) holds, then $\sigma_{l(a,a)} = \sigma_e$ implies $a^2 = (a \cdot a)l(a, a) = a \cdot a \in B$.

3. For $a, b \in B$, we have $aba = (a \cdot (b \cdot a))l(a, b \cdot a)l(b, a) \in B$, and thus $l(a, b \cdot a)l(b, a) = e$. By (3.16), $\lambda_{a,b\cdot a} \circ \lambda_{b,a} = \iota$, which is (2.15). Conversely, if $G = BH$ is reduced and (2.15) holds, then $\sigma_{l(a,b\cdot a)l(b,a)} = \sigma_e$ implies $aba = (a \cdot (b \cdot a))l(a, b \cdot a)l(b, a) = a \cdot (b \cdot a) \in B$.

4. For $a, b \in B$, we have $a^2 b = (a \cdot (a \cdot b))l(a, a \cdot b)l(a, b) \in B$, and thus $l(a, a \cdot b)l(a, b) = e$. By (3.16), $\lambda_{a,a\cdot b} \circ \lambda_{a,b} = \iota$, which is (2.17). Conversely, if $G = BH$ is reduced and (2.17) holds, then $\sigma_{l(a,a\cdot b)l(a,b)} = \sigma_e$ implies $a^2 b = (a \cdot (a \cdot b))l(a, a \cdot b)l(a, b) = a \cdot (a \cdot b) \in B$.

5. If (G-psA$_l$) holds, then by Lemma 3.9(1), (3.16), (3.17) and Proposition 2.2(3), we have that $\sigma(H)$ is a subgroup of psAut(B), which implies $(B, \cdot)$ is a pseudo-A$_l$ left loop. Conversely, assume $(B, \cdot)$ is a pseudo-A$_l$ left loop and that $G = BH$ is reduced. Fix $a, b \in B$ and let c be a companion of $\lambda_{a,b}$. For $x \in B$, we have $\sigma_{m(x,l(a,b))} = \sigma_{l(c,\lambda_{a,b}(x))^{-1}}$ by (3.16) and Proposition 2.2(3). Thus $m(x, l(a, b)) = l(c, \lambda_{a,b}(x))^{-1}$, and by Lemma 3.9(1), we have $cl(a, b)xl(a, b)^{-1} \in B$. Since H_0 is generated by elements of the form $l(a, b)$, $a, b \in B$, (G-psA$_l$)$_0$ follows.

6. If (G-A$_l$) holds, then by Lemma 3.9(2), (3.17) and Proposition 2.2(4), we have that $\sigma(H)$ is a subgroup of Aut(H), which implies $(B, \cdot)$ is an A$_l$ left loop. Conversely, assume $(B, \cdot)$ is an A$_l$ left loop and that $G = BH$ is reduced. Fix $a, b \in B$. For $x \in B$, we have $\sigma_{m(x,l(a,b))} = \sigma_e$ by (3.16) and Proposition 2.2(4). Thus $m(x, l(a, b)) = e$, and by Lemma 3.9(2), we have $l(a, b)xl(a, b)^{-1} \in B$. As with the preceding part, (G-A$_l$) follows. ∎

Corollary 3.11. *Let B be a left loop.*

1. *B satisfies LIP if and only if, for all $a \in B$, $L_a^{-1} \in \hat{B}$.*
2. *B satisfies LAP if and only if, for all $a \in B$, $L_a^2 \in \hat{B}$.*
3. *B is a Bol loop if and only if, for all $a, b \in B$, $L_a \circ L_b \circ L_a \in \hat{B}$.*
4. *B is an LC left loop if and only if, for all $a, b \in B$, $L_a \circ L_a \circ L_b \in \hat{B}$.*

Proof. Apply Theorem 3.10 to the reduced transversal decomposition $\mathrm{Sym}(B) = \hat{B}\,\mathrm{Rot}(B)$. ∎

Remark 3.12. Let G be a group. A subset $B \subseteq G$ is said to be a *twisted subgroup* of G if $e \in B$ and if $aBa \subseteq B$ for all $a \in B$ [1] [7]. In this jargon, we can restate Theorem 3.10(3) as follows: *If $G = BH$ be a reduced transversal decomposition, then B is a twisted subgroup if and only if $(B, \cdot)$ is a Bol loop.* This generalizes [7], Thm. 3.8.

Proposition 3.13. *Let $G = BH$ be a transversal decomposition.*

1. *If (G-Bol) holds, then (G-LIP) and (G-LAP) hold.*
2. *If (G-LC) holds, then (G-LIP) and (G-LAP) hold.*

Proof. 1. Fix $a \in B$. We have

$$B \ni a'aa' = a'(a \cdot a')l(a, a') = a'l(a, a').$$

Thus $l(a, a') = e$, and so $aa' = e$, i.e., $a^{-1} = a' \in B$, which establishes (G-LIP). We also have $a^2 = a1a \in B$, so that (G-LIP) holds.

2. Fix $a \in B$. We have

$$B \ni aaa' = a(a \cdot a')l(a, a') = al(a, a').$$

Thus $l(a, a') = e$, and so $aa' = e$, i.e., $a^{-1} = a' \in B$, which establishes (G-LIP). We also have $a^2 = aa1 \in B$, so that (G-LIP) holds. ∎

Proposition 3.14. *Let $G = BH$ be a transversal decomposition.*

1. *If (G-LIP) holds, then $l(a, b)^{-1} = l(a^{-1}, a \cdot b)$ for all $a, b \in B$.*
2. *If (G-LIP) and (G-A_l) hold, then $l(a, b)^{-1} = l(b^{-1}, a^{-1})$ for all $a, b \in B$.*
3. *If (G-Bol) holds, then $l(a, b)^{-1} = l(b, a)$ for all $a, b \in B$.*
4. *If (G-Bol) and (G-A_l) hold, then $l(a^{-1}, b^{-1}) = l(a, b)$ for all $a, b \in B$.*

Proof. 1. Using (LIP), we compute

$$\begin{aligned} l(a,b)l(a^{-1}, a \cdot b) &= (a \cdot b)^{-1}ab(a^{-1} \cdot (a \cdot b))^{-1}a^{-1}(a \cdot b) \\ &= (a \cdot b)^{-1}abb^{-1}a^{-1}(a \cdot b) = e. \end{aligned}$$

2. We have

$$l(a,b)b^{-1}a^{-1} = (a \cdot b)^{-1}abb^{-1}a^{-1} = (a \cdot b)^{-1} \in B.$$

Thus

$$B \ni l(a,b)^{-1}l(a,b)b^{-1}a^{-1}l(a,b) = (b^{-1} \cdot a^{-1})l(b^{-1}, a^{-1})l(a,b).$$

Therefore $l(b^{-1}, a^{-1})l(a, b) = e$.

3. We have $ab(b \cdot a)^{-1}ba \in B$ for $a, b \in B$ by Proposition 3.13(1). Now

$$\begin{aligned} ab(b \cdot a)^{-1}ba &= (a \cdot b)l(a,b)(b \cdot a)^{-1}(b \cdot a)l(b,a) \\ &= (a \cdot b)l(a,b)l(b,a). \end{aligned}$$

Thus $l(a, b)l(b, a) = e$.

4. This follows immediately from (2) and (3) and Proposition 3.13(1). ∎

Proposition 3.15. *Let $G = BH$ be a transversal decomposition satisfying (G-Bol). Then for all $a, b, x \in B$,*

$$((a \cdot b) \cdot (a^{-1} \cdot b^{-1}))l(a,b)xl(a,b)^{-1} \in B \tag{3.24}$$

In particular, (G-psA_l)$_0$ holds.

Note that Proposition 3.13(1) implies that (3.24) is well-defined.

Proof. For $a, b, x \in B$,

$$\begin{aligned} l(a,b)xl(a,b)^{-1} &= (a \cdot b)^{-1}abxb^{-1}a^{-1}(a \cdot b) \\ &= [(a \cdot b)^{-1}ab^2a(a \cdot b)^{-1}][(a \cdot b)a^{-1}b^{-1}xb^{-1}a^{-1}(a \cdot b)]. \end{aligned}$$

By (G-Bol), each of the expressions in square brackets is an element of B. Now since $a^{-1}b^{-2}a^{-1} \in B$, we have $a^{-1}b^{-2}a^{-1} = a^{-1} \cdot (b^{-2} \cdot a^{-1})$. Thus

$$\begin{aligned} (a \cdot b)^{-1}ab^2a(a \cdot b)^{-1} &= [(a \cdot b)a^{-1}b^{-2}a^{-1}(a \cdot b)]^{-1} \\ &= [(a \cdot b)(a^{-1} \cdot (b^{-2} \cdot a^{-1}))(a \cdot b)]^{-1} \\ &= [(a \cdot b) \cdot ((a^{-1} \cdot (b^{-2} \cdot a^{-1})) \cdot (a \cdot b))]^{-1}. \end{aligned}$$

Since B is a Bol loop (by Theorem 3.10(3)),

$$(a \cdot b)^{-1}ab^2a(a \cdot b)^{-1} = [(a \cdot b) \cdot (a^{-1} \cdot (b^{-2} \cdot (a^{-1} \cdot (a \cdot b))]^{-1}.$$

Applying LIP, LAP, and LIP once more, we obtain

$$(a \cdot b)^{-1}ab^2a(a \cdot b)^{-1} = [(a \cdot b) \cdot (a^{-1} \cdot b^{-1})]^{-1}.$$

Therefore

$$((a\cdot b)\cdot(a^{-1}\cdot b^{-1}))l(a,b)xl(a,b)^{-1}=(a\cdot b)a^{-1}b^{-1}xb^{-1}a^{-1}(a\cdot b)\in B,$$

as claimed. Since H_0 is generated by elements of the form $l(a,b)$, we have (G-psA$_l$)$_0$. ∎

Proposition 3.16. *Let $G = BH$ be a transversal decomposition.*

1. *(G-Br) holds if and only if the Bruck identity (2.19) holds and for all $a, b \in B$,*
$$m(b\cdot a, l(a,b))l(a,b)l(b,a)=e.$$
2. *Assume (G-A$_l$) holds. Then (G-Br) holds if and only if the Bruck identity (2.19) holds and $l(a,b)^{-1}=l(b,a)$ for all $a,b\in B$.*
3. *Assume (G-LIP) holds. Then (G-Br) holds if and only if AIP holds and $l(a,b)=l(a^{-1},b^{-1})$ for all $a,b\in B$.*
4. *Assume (G-Bol) holds. Then (G-Br) holds if and only if the Bruck identity (2.19) holds and $m(b\cdot a,l(a,b))=e$ for all $a,b\in B$.*
5. *Assume (G-Bol) and (G-A$_l$) hold. Then (G-Br), AIP, and the Bruck identity (2.19) are equivalent.*

Proof. Let $a, b \in B$ be given. Then $ab^2a = (a\cdot b)^2$ if and only if

$$l(a,b)ba = a\cdot b. \tag{3.25}$$

1. (3.25) is equivalent to
$$\begin{aligned} a\cdot b &= l(a,b)(b\cdot a)l(a,b)^{-1}l(a,b)l(b,a)\\ &= \lambda_{a,b}(b\cdot a)m(b\cdot a,l(a,b))l(a,b)l(b,a). \end{aligned}$$

Matching components, we find that this is equivalent to $a\cdot b=\lambda_{a,b}(b\cdot a)$, which is another form of (2.19), and $m(b\cdot a,l(a,b))l(a,b)l(b,a)=e$
as claimed.

2. This follows immediately from (1) since $m(b\cdot a,l(a,b))=e$ by (G-A$_l$) and Lemma 3.9(2).

3. (3.25) holds if and only if
$$l(a,b)=(a\cdot b)a^{-1}b^{-1}=(a\cdot b)(a^{-1}\cdot b^{-1})l(a^{-1},b^{-1}).$$

Matching components, this is equivalent to $(a\cdot b)(a^{-1}\cdot b^{-1})=1$ and $l(a,b)=l(a^{-1},b^{-1})$ as claimed.

4. This follows from (1) and Proposition 3.14(3).

5. This follows from (3), (4), and Proposition 3.14(4). ∎

Kikkawa [17], Prop. 1.13(4), has shown that in a homogeneous left loop, AIP implies $\lambda_{a,b}=\lambda_{a^{-1},b^{-1}}$ for all $a,b\in B$. Putting this result together with Propositions 3.14(2), 3.16(2) and 3.16(3) implies the following.

Proposition 3.17. *Let $G = BH$ be a* reduced *transversal decomposition satisfying (G-LIP) and (G-A$_l$). Then (G-Br), AIP, and the Bruck identity (2.19) are equivalent.*

Remark 3.18. Let $G = BH$ be a transversal decomposition satisfying (G-LAP) and (G-Br). Then for all $a \in B$, $aB^2a \subseteq B$. Assume that the mapping $x \mapsto x^2$

is surjective. Then $B^2 = B$, and thus (G-Bol) holds. If $x \mapsto x^2$ is a permutation, then the loop operation in B is given by

$$a \cdot b = (ab^2a)^{1/2}. \tag{3.26}$$

In this case, $(B, \cdot)$ is a B-loop. It is interesting to compare the operation (3.26) in B with the operation discussed by Glauberman [9] and Kikkawa [17] (Ex. 1.5), namely, $a * b = a^{1/2}ba^{1/2}$. We have $a \cdot b = (a^2 * b^2)^{1/2}$. Thus $a * b = [a^{1/2}b^{1/2}]_B^2$. The operation $a \cdot b = [ab]_B$ of the present paper is somewhat more natural with respect to the group structure $G = BH$.

Let $G = BH$ be a transversal decomposition satisfying (G-LIP). We define a mapping $\tau : G \to G$ by $\tau(ah) = a^{-1}h$. Then τ is involutive, that is, $\tau \circ \tau = \iota_G$. We now interpret some of the preceding results in terms of τ (cf. [17], Thm. 6.1).

Proposition 3.19. *Let $G = BH$ be a transversal decomposition satisfying (G-LIP). For all $g \in G$, $gB\tau(g)^{-1} \subseteq B$ if and only if both (G-Bol) and (G-A$_l$) hold.*

Proof. Trivial. ∎

Theorem 3.20. *Let $G = BH$ be a transversal decomposition satisfying (G-LIP). Then $\tau : G \to G$ is an automorphism if and only if each of the following is satisfied.*

1. *(G-Br) holds.*
2. *$\sigma_h(a^{-1}) = \sigma_h(a)^{-1}$ for all $a \in B$, $h \in H$.*
3. *$m(a^{-1}, h) = m(a, h)$ for all $a \in B$, $h \in H$.*

Proof. On one hand, we compute

$$\begin{aligned} \tau(ahbk) &= \tau((a \cdot \sigma_h(b))l(a, \sigma_h(b))m(b, h)k) \\ &= (a \cdot \sigma_h(b))^{-1}l(a, \sigma_h(b))m(b, h)k. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \tau(ah)\tau(bk) &= a^{-1}hb^{-1}k \\ &= (a^{-1} \cdot \sigma_h(b^{-1}))l(a^{-1}, \sigma_h(b^{-1}))m(b^{-1}, h)k \end{aligned}$$

Now assume τ is an automorphism. Matching components yields

$$(a \cdot \sigma_h(b))^{-1} = a^{-1} \cdot \sigma_h(b^{-1}) \tag{3.27}$$

$$l(a, \sigma_h(b))m(b, h) = l(a^{-1}, \sigma_h(b^{-1}))m(b^{-1}, h). \tag{3.28}$$

Taking $h = e$ in (3.27) gives AIP, while taking $a = 1$ gives (2). Taking $h = e$ in (3.28) (and using (3.14) gives $l(a, b) = l(a^{-1}, b^{-1})$. By Proposition 3.16(2), (1) holds. Taking $a = 1$ in (3.28) gives (3). Conversely, (1), (2), and (3) together imply (3.27) and (3.28) (using Proposition 3.16(2)), which in turn implies τ is an automorphism. ∎

Corollary 3.21. *Let $G = BH$ be a transversal decomposition satisfying (G-LIP) and (G-A$_l$). Then $\tau : G \to G$ is an automorphism if and only if (G-Br) holds.*

Proof. Conditions (2) and (3) of Theorem 3.20 are trivial since (G-A$_l$) holds, using Lemma 3.9. ∎

Remark 3.22. Automorphisms or, equivalently, anti-automorphisms of order 2 have been used by various authors to study (in the language of the present paper) transversal decompositions satisfying (G-Br), (G-Bol) and (G-A$_l$), that is, Bruck loops and B-loops. See, for instance, Foguel and Ungar [7], Glauberman [9], Im [12], Karzel and Wefelscheid [15], and Kikkawa [17].

We now consider some examples of internal semidirect products.

Example 3.23. (*Polar Decomposition*) Let $GL(n,\mathbb{C})$ denote the general linear group of $n \times n$ complex invertible matrices, let $\mathcal{P}(n)$ denote the subset of all $n \times n$ positive definite Hermitian matrices, and let $U(n)$ denote the subgroup of unitary $n \times n$ complex matrices. The *polar decomposition* asserts that every $M \in GL(n,\mathbb{C})$ can be uniquely factored as $M = AU$ for a unique $A = (MM^*)^{1/2} \in \mathcal{P}(n)$ and $U = (MM^*)^{-1/2}A \in U(n)$, where M^* is the conjugate transpose of M and where the unique positive definite square root of MM^* is intended. Then the polar decomposition is a transversal decomposition

$$GL(n,\mathbb{C}) = \mathcal{P}(n) \cdot U(n). \tag{3.29}$$

The induced binary operation (3.5), denoted here by $\odot$, is given by

$$A \odot B = (AB(AB)^*)^{1/2} = (AB^2A)^{1/2} \tag{3.30}$$

for $A, B \in \mathcal{P}(n)$; compare with 3.26. The transversal mapping $l : \mathcal{P}(n) \times \mathcal{P}(n) \to U(n)$ is given by

$$l(A,B) = (AB^2A)^{-1/2}AB \tag{3.31}$$

for $A, B \in \mathcal{P}(n)$. We have $\mathcal{P}(n)^{-1} \subseteq \mathcal{P}(n)$, i.e., (G-LIP) holds. Thus the involution $\tau : GL(n,\mathbb{C}) \to GL(n,\mathbb{C}) : AU \longmapsto A^{-1}U$ is well-defined. This mapping is explicitly given by

$$\begin{aligned}
\tau(M) &= \tau((MM^*)^{1/2}(MM^*)^{-1/2}M) \\
&= (MM^*)^{-1/2}(MM^*)^{-1/2}M \\
&= (MM^*)^{-1}M \\
&= (M^*)^{-1}
\end{aligned}$$

for $M \in GL(n,\mathbb{C})$. For $A \in \mathcal{P}(n)$, $M \in GL(n,\mathbb{C})$, we have $MA\tau(M)^{-1} = MAM^* \in \mathcal{P}(n)$. By Proposition 3.19, (G-Bol) and (G-A$_l$) hold. Since τ is the composition of two involutory anti-automorphisms (conjugate transposition and inversion), τ is an automorphism. By Corollary 3.21, (G-Br) holds. In addition, the squaring mapping is a permutation of $\mathcal{P}(n)$. Putting these facts together, we have that $(\mathcal{P}(n), \oplus)$ is a B-loop.

Example 3.24. (*Subgroups of* $GL(n,\mathbb{C})$) Subgroups of $GL(n,\mathbb{C})$ respecting the polar decomposition include the special linear group $SL(n,\mathbb{C})$, the real general linear group $GL(n,\mathbb{R})$, the group $U(m,n)$, and the complex symplectic group $Sp(n,\mathbb{C})$. Taking intersections of these yields more such groups. Here we limit ourselves to one specific example: The group $SU(1,1)$ consists of those 2×2 complex matrices preserving the form $|z_1|^2 - |z_2|^2$ on $\mathbb{C}^2$ which also have determinant 1. The polar decomposition of this group is

$$SU(1,1) = \mathcal{P}U(1,1) \cdot S(U(1) \times U(1)).$$

where $S(U(1)\times U(1))$ is the subgroup of matrices of the form $\begin{pmatrix} \bar{a} & 0 \\ 0 & a \end{pmatrix}$, $a \in S^1$, and $\mathcal{P}U(1,1) = \mathcal{P}(n) \cap U(1,1)$ is the set of positive definite Hermitian matrices in $U(1,1)$. Thus $(\mathcal{P}U(1,1), \oplus)$ is a subloop of the B-loop $(\mathcal{P}(2), \oplus)$ of Example 3.23. Note that the mapping $Q : S^1 \to S(U(1)\times U(1))$ defined by $Q(a) = \begin{pmatrix} \bar{a} & 0 \\ 0 & a \end{pmatrix}$ is an isomorphism of groups. A matrix L in $\mathcal{P}U(1,1)$ can be parametrized by a number $z \in \mathbb{D}$ as follows:

$$L = L(z) = \gamma_z \begin{pmatrix} 1 & \bar{z} \\ z & 1 \end{pmatrix}$$

where $\gamma_z = (1-|z|^2)^{-1/2}$. The mapping $L : \mathbb{D} \to \mathcal{P}U(1,1)$ turns out to be an isomorphism from the B-loop $(\mathbb{D}, \oplus)$ of Example 2.10 to $(\mathcal{P}U(1,1), \oplus)$.

Remark 3.25. The polar decompositions of Examples 3.23 and 3.24 are special cases of the global Cartan decomposition of a Lie group associated with a Riemannian symmetric space of noncompact type [11]. Any such space, realized as a subset of the Lie group, can be given the structure of a B-loop; this actually follows quite easily from the results herein. For the Hermitian case (bounded symmetric domains), the result was shown in [8], while the general Riemannian case was worked out in [19] and [22]. For related work over Pythagorean fields, see [16].

Example 3.26. We will show that the group $SU(1,1)$ has a different transversal decomposition from that of Example 3.24, leading to a different internal semidirect product structure. With notation as in that example, let $A = L(z)Q(a)$, $z \in \mathbb{D}$, $a \in S^1$, be the polar decomposition of a given $A \in SU(1,1)$. Let

$$R(a,z) = \bar{a}L(z) = \bar{a}\gamma_z \begin{pmatrix} 1 & \bar{z} \\ z & 1 \end{pmatrix},$$

and let

$$S^1 \cdot \mathcal{P}U(1,1) = \{R(a,z) : a \in S^1,\ z \in \mathbb{D}\}.$$

Let

$$T(a) = aQ(a) = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix},$$

and note that $T(a) \in \{1\} \times U(1)$. Then $A = R(a,z)T(a)$. Thus we have shown that $SU(1,1)$ decomposes as follows:

$$SU(1,1) = (S^1 \cdot \mathcal{P}U(1,1)) \cdot (\{1\} \times U(1)).$$

This is a transversal decomposition; the uniqueness follows from the uniqueness of the polar decomposition. It is straightforward to show that the induced binary operation and corresponding transversal mapping on $S^1 \cdot \mathcal{P}U(1,1)$ are given by

$$R(a,z) \odot R(b,w) = R\left(ab\frac{1+\bar{z}w}{|1+\bar{z}w|}, z \oplus w\right)$$

and

$$l(R(a,z), R(b,w)) = T\left(\frac{1+\bar{z}w}{1+z\bar{w}}\right),$$

respectively, where $\oplus$ is given by (2.22). Straightforward computations give

$$T(a)R(b,z)T(a)^{-1} = R(b,a^2z)$$

and

$$R(a,z)R(b,w)R(a,z) = R(ab^2c/|c|, (z\oplus w)\oplus z),$$

where $c = 1+\bar{z}w+z\bar{w}+|z|^2$. Thus (G-A$_l$) and (G-Bol) hold, from which it follows that $(S^1\cdot\mathcal{P}U(1,1),\odot)$ is a homogeneous Bol loop. It is easy to show directly that $S^1\cdot\mathcal{P}U(1,1)$ does not satisfy AIP and hence is not a Bruck loop.

The loop $(S^1\cdot\mathcal{P}U(1,1),\odot)$ is isomorphic to a loop structure on the complex hyperbolic plane $H_{\mathbb{C}} = \left\{(x_0,x_1)^t\in\mathbb{C}^2 : |x_0|^2-|x_1|^2=1\right\}$. For $(x_0,x_1)^t,(y_0,y_1)^t\in H_{\mathbb{C}}$, define

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \odot \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{x_0}{\bar{x}_0}(\bar{x}_0y_0+\bar{x}_1y_1) \\ x_0y_1+x_1y_0 \end{pmatrix}.$$

Then $(H_{\mathbb{C}},\odot)$ is a loop. (This is a simplified version of an example in [24].) The following sequence of loop homomorphisms is exact:

$$1\to S^1 \xrightarrow{\alpha} H_{\mathbb{C}} \xrightarrow{\pi} \mathbb{D}\to 0,$$

where $\alpha(z) = (z,0)^t$ for $z\in S^1$ and $\pi\left((x_0,x_1)^t\right) = x_1/x_0$. In fact, $H_{\mathbb{C}}$ is a central, invariant extension of $\mathbb{D}$ by S^1 [22]. The mapping $(H_{\mathbb{C}},\odot)\to(S^1\cdot PU(1,1),\odot)$: $(x_0,x_1)\mapsto R(x_0/|x_0|, x_1/x_0)$ is an isomorphism of loops.

Example 3.27. (*Projective Groups*) Let $G\le SL(n,\mathbb{C})$ be a subgroup respecting the polar decomposition. Let $B = G\cap\mathcal{P}(n)$ and $H = G\cap U(n)$. Then $G = BH$ is a transversal decomposition. The kernel of the conjugation homomorphism $U\mapsto\sigma_U$, where $\sigma_U(A) = UAU^*$ for $A\in B$, $U\in H$, is the group $\ker(\sigma) = G\cap\{cI : c\in\mathbb{C}\}$ of scalar matrices in G. Thus $PG = G/\ker(\sigma)$ is the *projective group* associated to G, and similarly define PH. Applying (3.22) to the present setting, we have the reduced transversal decomposition

$$PG = B\cdot PH.$$

We will refer to this as a *projective polar decomposition.*

As a specific example, consider the Möbius group $PSU(1,1)$. The projective polar decomposition of this group is

$$PSU(1,1) = \mathcal{P}U(1,1)\cdot PS(U(1)\times U(1)).$$

The only scalar matrices in $S(U(1)\times U(1))$ are $\pm I$, and thus $PS(U(1)\times U(1)) = S(U(1)\times U(1))/\{\pm I\}$. In terms of (3.20) and (3.21), the exact sequence of groups

$$1\to\{\pm I\}\to S(U(1)\times U(1))\to PS(U(1)\times U(1))\to 1$$

induces the exact sequence of semidirect product groups

$$1\to\{\pm I\}\to SU(1,1)\to PSU(1,1)\to 1.$$

Example 3.28. (*Upper Triangular Matrices*) Let $\mathbb{F}$ be a field containing $1/2$. For $x,y,z\in\mathbb{F}$, let

$$T(x,y,z) = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

and let $\mathcal{T}(3,\mathbb{F}) = \{T(x,y,z) : x,y,z \in \mathbb{F}\}$ be the group of 3×3 strictly upper triangular matrices over $\mathbb{F}$. For $x_1, x_2 \in \mathbb{F}$, let

$$A(x_1,x_2) = \begin{pmatrix} 1 & x_1 & \frac{1}{2}x_1x_2 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix},$$

and let $A(3,\mathbb{F}) = \{A(x_1,x_2) : x_1,x_2 \in \mathbb{F}\}$. For $c \in \mathbb{F}$, let

$$M(c) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and let $M(3,\mathbb{F}) = \{M(c) : c \in \mathbb{F}\}$. Then $M(3,\mathbb{F})$ is a subgroup of $\mathcal{T}(3,\mathbb{F})$. An arbitrary matrix $T(x,y,z) \in \mathcal{T}(3,\mathbb{F})$ factors as follows:

$$T(x,y,z) = A(x,z)M\left(y - xz/2\right).$$

It is easy to show by direct computation that this factorization is unique. In addition, $A(x,z) = M(y-xz/2)$ if and only if $x=y=z=0$, which implies $A(3,\mathbb{F}) \cap M(3,\mathbb{F}) = \{I\}$. Thus

$$\mathcal{T}(3,\mathbb{F}) = A(3,\mathbb{F}) \cdot M(3,\mathbb{F}) \tag{3.32}$$

is a transversal decomposition. Denote the induced binary operation (3.5) on $A(3,\mathbb{F})$ by $\oplus$. Then $\oplus$ is given by

$$A(x_1,x_2) \oplus A(y_1,y_2) = A(x_1+y_1, x_2+y_2). \tag{3.33}$$

It is immediate that $A(3,\mathbb{F})$ is an abelian group isomorphic to $\mathbb{F}^2$. The transversal mapping $l : A(3,\mathbb{F}) \times A(3,\mathbb{F}) \to M(3,\mathbb{F})$ is given by

$$l(A(x_1,x_2), A(y_1,y_2)) = M((x_1y_2 - x_2y_1)/2). \tag{3.34}$$

Since this is nontrivial, $A(3,\mathbb{F})$ is not a subgroup of $\mathcal{T}(3,\mathbb{F})$. The subgroup $M(3,\mathbb{F})$ is the center of $\mathcal{T}(3,\mathbb{F})$, and thus trivially normalizes $A(3,\mathbb{F})$. Thus $\mathcal{T}(3,\mathbb{F})$ is an internal semidirect product of the abelian group $(A(3,\mathbb{F}),\oplus)$ with the abelian group $M(3,\mathbb{F})$. However, as noted in Remark 3.7(4), this is *not* the usual internal semidirect product of groups.

This example turns out to generalize to $\mathcal{T}(n,\mathbb{F})$ for any $n \geq 3$. For $n > 3$, $(A(n,\mathbb{F}),\oplus)$ is a loop but not a group, and the homomorphism $\sigma : M(n,\mathbb{F}) \to \mathrm{Aut}(A(n,\mathbb{F}))$ is nontrivial.

Remark 3.29. Examples 3.27 and 3.28 are counterexamples to a claim of Sabinin ([23], Thm. 8) that (in the notation and terminology of the present paper) $\sigma : H \to \mathrm{Rot}(B)$ is always injective.

4. External Semidirect Products

In this section we generalize the standard semidirect product $B \rtimes H$ to the case where H is not necessarily a rotary-closed subgroup of $\mathrm{Rot}(B)$, but rather there is a homomorphism $\sigma : H \to \mathrm{Rot}(B)$. Our discussion will show that, in a certain sense, our definition of external semidirect product is the optimal one.

Let B be a set with a distinguished element 1 and let H be a group. Assume that $B \times H$ has a binary operation which makes it a group satisfying the following properties:

(E1) $\{1\} \times H$ is a subgroup isomorphic to H.

(E2) $(a,h) = (a,e)(1,h)$ for all $a \in B$, $h \in H$.

Then $B \times H = (B \times \{e\})(\{1\} \times H)$ is a transversal decomposition. Indeed, if $(a,h) = (b,k)$ for some $a,b \in B$, $h,k \in H$, then using (E2) and then (E1) gives $(a,e) = (b,kh^{-1})$, which implies $a = b$ and $h = k$. Making the usual identifications $B \cong B \times \{e\}$ and $H \cong \{1\} \times H$, we have an induced operation $\cdot$ on B and an induced transversal mapping $l : B \times B \to H$, both defined by

$$(a,e)(b,e) = (a \cdot b, l(a,b))$$

for $a,b \in B$. We also have a mapping $m : B \times H \to H$ and (what turns out to be) a homomorphism $\sigma : H \to \mathrm{Rot}(B)$, both defined by

$$(1,h)(a,1)(1,h^{-1}) = (\sigma_h(a), m(a,h))$$

for $a \in B$, $h \in H$. Thus given arbitrary elements $(a,h),(b,k) \in B \times H$, we compute

$$\begin{aligned} (a,h)(b,k) &= (a,e)(1,h)(b,e)(1,h^{-1})(1,h)(1,k) \\ &= (a,e)(\sigma_h(b),e)(1,m(b,h))(1,hk) \\ &= (a \cdot \sigma_h(b), l(a,\sigma_h(b)))(1,m(b,h)hk) \\ &= (a \cdot \sigma_h(b), l(a,\sigma_h(b))m(b,h)h^{-1}k). \end{aligned}$$

We will use this to construct our definition. Starting over, let B be a left loop and let H be a group. Assume there exist a mapping $l : B \times B \to H$, a mapping $m : B \times H \to H$, and a homomorphism $\sigma : H \to \mathrm{Rot}(B)$. With Theorem 3.3 as motivation, assume the following conditions hold:

(S1) For all $a,b \in B$,

$$\sigma_{l(a,b)} = \lambda_{a,b}.$$

(S2) For all $a \in B$, $h \in H$,

$$\sigma_{m(a,h)} = \mu_a(h).$$

Define a binary operation on $B \times H$ by

$$(a,h)(b,k) = (a \cdot \sigma_h(b), l(a,\sigma_h(b))m(b,h)hk). \tag{4.1}$$

The question is thus: what are the minimal additional assumptions necessary for $B \times H$ with the product given by (4.1) to be a group?

If $B \times H$ is a group and both (E1) and (E2) hold, then $B \times H = (B \times \{e\})(\{1\} \times H)$ is a transversal decomposition. Thus there is an induced product on B, which we will denote by $\hat{\cdot}$, an induced transversal mapping $\hat{l} : B \times B \to H$, an induced mapping $\hat{m} : B \times H \to H$, and an induced homomorphism $\hat{\sigma} : H \to \mathrm{Rot}(B)$. Using (4.1), (E1) and (E2), we compute

$$\begin{aligned} (a \hat{\cdot} b, \hat{l}(a,b)) &= (a,e)(b,e) \\ &= (a \cdot b, l(a,b)m(b,e)) \end{aligned}$$

and

$$\begin{aligned} (\hat{\sigma}_h(a), \hat{m}(a,h)) &= (1,h)(a,e)(1,h^{-1}) \\ &= (1,h)(a,h^{-1}) \\ &= (\sigma_h(a), l(1,\sigma_h(a))m(a,h)). \end{aligned}$$

Thus we see that $\hat{\cdot} = \cdot$ and $\hat{\sigma} = \sigma$. In addition, we have

$$\hat{l}(a,b) = l(a,b)m(b,e) \tag{4.2}$$

$$\hat{m}(a,h) = l(1,\sigma_h(a))m(a,h) \tag{4.3}$$

for all $a,b \in B$, $h \in H$. If we assume that $\hat{l} = l$ and $\hat{m} = m$, then we have the following necessary requirements:

(S3) For all $a \in B$,

$$l(1,a) = e.$$

(S4) For all $a \in B$,

$$m(a,e) = e.$$

Taking $b = 1$ in (4.2), applying (3.8) to $\hat{l}(a,1)$, and using (S4), we obtain

(S5) For all $a \in B$,

$$l(a,1) = e.$$

Taking $a = 1$ in (4.3), applying (3.15) to $\hat{m}(1,h)$, and using (S3), we obtain

(S6) For all $h \in H$,

$$m(1,h) = e.$$

Next we consider the group axioms which must be satisfied by $B \times H$. We have

$$\begin{aligned}(a,h)(1,e) &= (a \cdot \sigma_h(1), l(a,\sigma_h(1))m(1,h)h) \\ &= (a,h)\end{aligned}$$

by (S5) and (S6), and

$$\begin{aligned}(1,e)(a,h) &= (1 \cdot \sigma_e(a), l(1,\sigma_e(a))m(a,e)h) \\ &= (a,h)\end{aligned}$$

by (S3) and (S4). Therefore, the hypotheses we have so far give us that $(1,e)$ is the identity element of $B \times H$.

Next, we impose associativity on $B \times H$. By computing an arbitrary product $(a,h)(b,k)(c,t)$ in two different ways, matching H-components (matching B-components gives no new information), and simplifying, we obtain the following technical condition which must be satisfied.

(TC) For all $a,b,c \in B$, $h,k \in H$,

$$\begin{aligned}&l(a \cdot \sigma_h(b), (\lambda_{a,\sigma_h(b)} \circ \mu_b(h) \circ \sigma_{hk})(c))m(c, l(a,\sigma_h(b))m(b,h)hk)l(a,\sigma_h(b))m(b,h) \\ &\quad = l(a, \sigma_h(b) \cdot (\mu_b(h) \circ \sigma_{hk})(c))m(b \cdot \sigma_k(c), h)hl(b,\sigma_k(c))m(c,k)h^{-1}.\end{aligned}$$

Fortunately, (TC) can be replaced by three simpler conditions to which it is equivalent. First, taking $h = k = e$ in (TC) and using (S4), we obtain

(S7) For all $a,b,c \in B$,

$$l(a \cdot b, \lambda_{a,b}(c))m(c, l(a,b))l(a,b) = l(a, b \cdot c)l(b,c).$$

Second, taking $a = b = 1$ in (TC) and using (S3) and (S6) (and writing a for c) we obtain

(S8) For all $a \in B$, $h,k \in H$,

$$m(c,hk) = m(\sigma_k(c),h)hm(c,k)h^{-1}.$$

Finally, taking $a = 1$, $k = e$ in (TC) and using (S3) and (S4) (and making the replacements $b \to a$, $c \to b$), we obtain

(S9) For all $a, b \in B$, $h \in H$,

$$l(\sigma_h(a), (\mu_a(h) \circ \sigma_h)(b)) m(b, m(a,h)h) m(a,h) = m(a \cdot b, h) h l(a,b) h^{-1}.$$

Thus we have shown one direction of the following.

Lemma 4.1. *Condition (TC) is equivalent to conditions (S7), (S8) and (S9).*

We omit the tedious proof that (S7), (S8) and (S9) imply (TC), except to say that starting with the left hand side of (TC), one can obtain the right hand side by two applications of (S8), then one application of (S7), and then one application of (S9).

Next we consider inverses in $B \times H$. Consider first an element of the form (a, e). If (x, h) is to be the right inverse of (a, e), then

$$\begin{aligned} (1, e) &= (a, e)(x, h) \\ &= (a \cdot x, l(a, x) m(x, e) h). \end{aligned}$$

This implies $x = a' = L_a^{-1}(1)$, and using (S4), $h = l(a, a')^{-1}$, so that

$$(a, e)^{-1} = (a', l(a, a')^{-1}). \tag{4.4}$$

Now we consider a general element $(a, h) \in B \times H$ and use (E2), (E1), (4.4) and (S3) to derive the inverse:

$$\begin{aligned} (a, h)^{-1} &= ((a, e)(1, h))^{-1} \\ &= (1, h^{-1})(a', l(a, a')^{-1}) \\ &= (\sigma_{h^{-1}}(a'), l(1, \sigma_{h^{-1}}(a')) m(a', h^{-1}) h^{-1} l(a, a')^{-1}) \\ &= (\sigma_{h^{-1}}(a'), m(a', h^{-1}) h^{-1} l(a, a')^{-1}). \end{aligned} \tag{4.5}$$

We check that this candidate is indeed a right inverse:

$$\begin{aligned} (1, e) &= (a, h)(\sigma_{h^{-1}}(a'), m(a', h^{-1}) h^{-1} l(a, a')^{-1}) \\ &= (a \cdot a', l(a, a') m(\sigma_{h^{-1}}(a'), h) h m(a', h^{-1}) h^{-1} l(a, a')^{-1}). \end{aligned}$$

By (S8), the H-component of the last step of this calculation indeed simplifies to e. Similar computations show that (4.5) is a left inverse, although it is not necessary to check this; a two-sided identity, right inverses and associativity are sufficient for $B \times H$ to be a group.

Definition 4.2. Let B be a left loop and let H be a group. Assume there exist a mapping $l : B \times B \to H$, a mapping $m : B \times H \to H$, and a homomorphism $\sigma : H \to \mathrm{Rot}(B)$ such that conditions (S1) through (S9) are satisfied. Define a binary operation $\cdot$ on the set $B \times H$ by

$$(a, h) \cdot (b, k) = (a \cdot \sigma_h(b), l(a, \sigma_h(b)) m(b, h) h k)$$

for $a, b \in B$, $h, k \in H$. Then $(B \times H, \cdot)$ is called the *external semidirect product* of B with H given by (σ, l, m), and is denoted $B \rtimes_{(\sigma,l,m)} H$.

Of course, the whole discussion leading up to this result was a sketch of the proof of the following.

Theorem 4.3. $(B \rtimes_{(\sigma,l,m)} H, \cdot)$ *is a group.*

Remark 4.4. As with the internal product, there are various special cases of the external product which are of interest.

1. If the transversal mapping $l : B \times B \to H$ can be chosen to be trivial, i.e., $l(a, b) = e$ for all $a, b \in B$, then B is a group (by (S1)). In this case, we have an external version of Jajcay's "rotary product" of groups [13].
2. If the mapping $m : B \times B \to H$ can be chosen to be trivial, i.e., $m(a, h) = e$ for all $a \in B$, $h \in H$, then B is an A_l-loop (by (S2)).
3. If both l and m can be chosen to be trivial, then $B \rtimes_{(\sigma,l,m)} H = B \rtimes_\sigma H$ is the usual external semidirect product of groups.
4. If $\sigma(H) = \{\iota\}$, then B is a group, but if $l : B \times B \to H$ is nontrivial, then B is not (isomorphic to) a subgroup of $B \rtimes_{(\sigma,l,m)} H$.

Our examples are homogeneous Bol loops. Thus we will choose m to be trivial, so that (S2), (S4), (S6) and (S8) are trivial. (S7) simplifies to

(S7') For all $a, b, c \in B$, $h \in H$,
$$l(a \cdot b, \lambda_{a,b}(c))l(a, b) = l(a, b \cdot c)l(b, c).$$

and (S9) simplifies to

(S9') For all $a, b \in B$, $h \in H$,
$$l(\sigma_h(a), \sigma_h(b)) = hl(a, b)h^{-1}.$$

Example 4.5. The mapping $\phi : S^1 \to S^1$ defined by $\phi(a) = a^2$ is a homomorphism of groups, and we consider the target copy of S^1 to be a subgroup of $\mathrm{Aut}(\mathbb{D}, \oplus)$. Define $l : \mathbb{D} \times \mathbb{D} \to S^1$ by
$$l(x, y) = \frac{1 + x\bar{y}}{|1 + \bar{x}y|}$$
for $x, y \in \mathbb{D}$. Then $\phi(l(x, y)) = \lambda_{x,y} = (1+x\bar{y})/(1+\bar{x}y)$, as in Example 2.10. Define $m : \mathbb{D} \times S^1 \to S^1$ trivially. It is easy to check that (S1), (S3), (S6), (S7') and (S9') are satisfied. Thus we have an external semidirect product $\mathbb{D} \rtimes_{(\phi,l)} S^1$. The exact sequence of groups
$$1 \to \{\pm 1\} \to S^1 \xrightarrow{\phi} S^1 \to 1$$
induces an exact sequence of semidirect product groups
$$1 \to \{\pm 1\} \to \mathbb{D} \rtimes_{(\phi,l)} S^1 \xrightarrow{\hat{\phi}} \mathbb{D} \rtimes S^1 \to 1$$
where $\hat{\phi}(z, a) = (z, \phi(a)) = (z, a^2)$. We now show the relationship between this construction and Example 3.27. Recall the isomorphism $Q : S^1 \to S(U(1) \times U(1))$ given by $Q(a) = \begin{pmatrix} \bar{a} & 0 \\ 0 & a \end{pmatrix}$. As the following diagram indicates, there exists an isomorphism $\hat{Q} : S^1 \to PS(U(1) \times U(1))$.
$$\begin{array}{ccccccccc} 1 & \to & \{\pm 1\} & \to & S^1 & \xrightarrow{\phi} & S^1 & \to & 1 \\ & & \downarrow & & \downarrow_Q & & \downarrow_{\hat{Q}} & & \\ 1 & \to & \{\pm I\} & \to & S(U(1) \times U(1)) & \to & PS(U(1) \times U(1)) & \to & 1 \end{array}$$
This is explicitly given by
$$\hat{Q}(a) = \begin{cases} \left[Q\left(\frac{1+a}{|1+a|}\right)\right] & \text{if } a \neq -1 \\ [Q(i)] & \text{if } a = -1 \end{cases}$$

where the square brackets denote equivalence classes. (Note that $((1+a)/|1+a|)^2 = a$.) On the other hand, the mapping $(z, a) \mapsto P(z)Q(a)$ gives an isomorphism from $\mathbb{D} \rtimes_\phi S^1$ to $SU(1,1) = \mathcal{P}U(1,1) \cdot S(U(1) \times U(1))$. This and the following diagram yield an isomorphism between $\mathbb{D} \rtimes S^1$ and $PSU(1,1)$:

$$\begin{array}{ccccccccc} 1 & \to & \{(0,\pm 1)\} & \to & \mathbb{D} \rtimes_{(\phi,l)} S^1 & \xrightarrow{\hat{\phi}} & \mathbb{D} \rtimes S^1 & \to & 1 \\ & & \downarrow & & \downarrow_{PQ} & & \downarrow_{P\hat{Q}} & & \\ 1 & \to & \{\pm I\} & \to & SU(1,1) & \to & PSU(1,1) & \to & 1 \end{array}$$

The isomorphism is explicitly given by $(z, a) \mapsto P(z)\hat{Q}(a)$.

Example 4.6. Let $\mathbb{F}$ be a field containing $1/2$. Let $P = \mathbb{F}^2$ with the operation of vector addition, and let $H = \mathbb{F}$ with the operation of addition. Define $l : \mathbb{F}^2 \times \mathbb{F}^2 \to \mathbb{F}$ by $l((x_1, x_2), (y_1, y_2)) = \frac{1}{2}(x_1y_2 - x_2y_1)$. Define $\phi : \mathbb{F} \to \mathrm{Aut}(\mathbb{F}^2)$ trivially: $\phi(c) = \iota$. Finally, define $m : \mathbb{F}^2 \times \mathrm{Aut}(\mathbb{F}^2) \to \mathrm{Aut}(\mathbb{F}^2)$ trivially. Then (S1), (S3), (S6), (S7') and (S9') are satisfied. We thus have an external semidirect product $\mathbb{F}^3 = \mathbb{F}^2 \rtimes_{(\phi,l)} \mathbb{F}$. The operation is given by

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1y_2 - x_2y_1)).$$

As in Remark 4.4(4), this external semidirect product of groups does not reduce to the usual one. Clearly this external semidirect product is isomorphic to the internal semidirect product of Example 3.28.

We conclude with a few remarks about the relationship of the external semidirect product with the standard and internal semidirect products. These naturally generalize the usual relationships between these products of groups.

First, if $G = BH$ is an internal semidirect product of a left loop $(B, \cdot)$ with the subgroup H, then the mapping $g = ah \mapsto (a, h)$ is clearly an isomorphism of BH with $B \rtimes_{(\sigma,l,m)} H$. On the other hand, by factoring a given external semidirect product as $B \rtimes_{(\sigma,d,m)} H \cong (B \times \{e\})(\{1\} \times H)$, it is easy to see that it is isomorphic to an internal semidirect product; this was essentially our starting point for deriving the definition of external semidirect product.

The standard semidirect product is, of course, a special case of the external semidirect product. On the other hand, if $B \rtimes_{(\sigma,l,m)} H$ is an external semidirect product, then (repeating the discussion which led to (3.21)) the natural mapping $\hat{\sigma} : B \rtimes_{(\sigma,d,m)} H \to B \rtimes \sigma(H) : (a, h) \mapsto (a, \sigma_h)$ is an epimorphism. We have $\ker(\hat{\sigma}) = \{1\} \times \ker(\sigma)$, and thus the exact sequence of groups

$$1 \to \ker(\sigma) \to H \to \sigma(H) \to 1$$

induces an exact sequence of semidirect product groups

$$1 \to \ker(\hat{\sigma}) \to B \rtimes_{(\sigma,l,m)} H \to B \rtimes \sigma(H) \to 1.$$

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